

# CONGRUENCES ON BICYCLIC EXTENSIONS OF A LINEARLY ORDERED GROUP

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**ABSTRACT.** In the paper we study inverse semigroups  $\mathcal{B}(G)$ ,  $\mathcal{B}^+(G)$ ,  $\overline{\mathcal{B}}(G)$  and  $\overline{\mathcal{B}}^+(G)$  which are generated by partial monotone injective translations of a positive cone of a linearly ordered group  $G$ . We describe Green's relations on the semigroups  $\mathcal{B}(G)$ ,  $\mathcal{B}^+(G)$ ,  $\overline{\mathcal{B}}(G)$  and  $\overline{\mathcal{B}}^+(G)$ , their bands and show that they are simple, and moreover the semigroups  $\mathcal{B}(G)$  and  $\mathcal{B}^+(G)$  are bisimple. We show that for a commutative linearly ordered group  $G$  all non-trivial congruences on the semigroup  $\mathcal{B}(G)$  (and  $\mathcal{B}^+(G)$ ) are group congruences if and only if the group  $G$  is archimedean. Also we describe the structure of group congruences on the semigroups  $\mathcal{B}(G)$ ,  $\mathcal{B}^+(G)$ ,  $\overline{\mathcal{B}}(G)$  and  $\overline{\mathcal{B}}^+(G)$ .

## 1. INTRODUCTION AND MAIN DEFINITIONS

In this article we shall follow the terminology of [7, 8, 14, 16, 20].

A *semigroup* is a non-empty set with a binary associative operation. A semigroup  $S$  is called *inverse* if for any  $x \in S$  there exists a unique  $y \in S$  such that  $x \cdot y \cdot x = x$  and  $y \cdot x \cdot y = y$ . Such an element  $y$  in  $S$  is called the *inverse* of  $x$  and denoted by  $x^{-1}$ . The map defined on an inverse semigroup  $S$  which maps every element  $x$  of  $S$  to its inverse  $x^{-1}$  is called the *inversion*.

If  $S$  is a semigroup, then we shall denote the subset of idempotents in  $S$  by  $E(S)$ . If  $S$  is an inverse semigroup, then  $E(S)$  is closed under multiplication and we shall refer to  $E(S)$  as the *band* of  $S$ . If the band  $E(S)$  is a non-empty subset of  $S$ , then the semigroup operation on  $S$  determines the following partial order  $\preceq$  on  $E(S)$ :  $e \preceq f$  if and only if  $ef = fe = e$ . This order is called the *natural partial order* on  $E(S)$ . A *semilattice* is a commutative semigroup of idempotents. A semilattice  $E$  is called *linearly ordered* or a *chain* if its natural order is a linear order.

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If  $\mathfrak{C}$  is an arbitrary congruence on a semigroup  $S$ , then we denote by  $\Phi_{\mathfrak{C}}: S \rightarrow S/\mathfrak{C}$  the natural homomorphisms from  $S$  onto the quotient semigroup  $S/\mathfrak{C}$ . Also we denote by  $\Omega_S$  and  $\Delta_S$  the *universal* and the *identity* congruences, respectively, on the semigroup  $S$ , i.e.,  $\Omega(S) = S \times S$  and  $\Delta(S) = \{(s, s) \mid s \in S\}$ . A congruence  $\mathfrak{C}$  on a semigroup  $S$  is called *non-trivial* if  $\mathfrak{C}$  is distinct from the universal and the identity congruence on  $S$ , and a *group congruence* if the quotient semigroup  $S/\mathfrak{C}$  is a group. Every inverse semigroup  $S$  admits a least group congruence  $\mathfrak{C}_{mg}$ :

$$a\mathfrak{C}_{mg}b \text{ if and only if there exists } e \in E(S) \text{ such that } ae = be$$

(see [20, Lemma III.5.2]).

A map  $h: S \rightarrow T$  from a semigroup  $S$  to a semigroup  $T$  is said to be an *antihomomorphism* if  $(a \cdot b)h = (b)h \cdot (a)h$ . A bijective antihomomorphism is called an *antiisomorphism*.

If  $S$  is a semigroup, then we shall denote by  $\mathcal{R}$ ,  $\mathcal{L}$ ,  $\mathcal{J}$ ,  $\mathcal{D}$  and  $\mathcal{H}$  the Green's relations on  $S$  (see [8]):

$$\begin{aligned} a\mathcal{R}b &\text{ if and only if } aS^1 = bS^1; \\ a\mathcal{L}b &\text{ if and only if } S^1a = S^1b; \\ a\mathcal{J}b &\text{ if and only if } S^1aS^1 = S^1bS^1; \\ \mathcal{D} &= \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}; \\ \mathcal{H} &= \mathcal{L} \cap \mathcal{R}. \end{aligned}$$

Let  $\mathcal{I}_X$  denote the set of all partial one-to-one transformations of an infinite set  $X$  together with the following semigroup operation:  $x(\alpha\beta) = (x\alpha)\beta$  if  $x \in \text{dom}(\alpha\beta) = \{y \in \text{dom } \alpha \mid y\alpha \in \text{dom } \beta\}$ , for  $\alpha, \beta \in \mathcal{I}_X$ . The semigroup  $\mathcal{I}_X$  is called the *symmetric inverse semigroup* over the set  $X$  (see [8]). The symmetric inverse semigroup was introduced by Wagner [21] and it plays a major role in the theory of semigroups.

The bicyclic semigroup  $\mathcal{C}(p, q)$  is the semigroup with the identity 1 generated by two elements  $p$  and  $q$  subjected only to the condition  $pq = 1$ . The distinct elements of  $\mathcal{C}(p, q)$  are exhibited in the following useful array

$$\begin{array}{cccccc} 1 & p & p^2 & p^3 & \cdots \\ q & qp & qp^2 & qp^3 & \cdots \\ q^2 & q^2p & q^2p^2 & q^2p^3 & \cdots \\ q^3 & q^3p & q^3p^2 & q^3p^3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

and the semigroup operation on  $\mathcal{C}(p, q)$  is determined as follows:

$$q^k p^l \cdot q^m p^n = q^{k+m-\min\{l,m\}} p^{l+n-\min\{l,m\}}.$$

The bicyclic semigroup plays an important role in the algebraic theory of semigroups and in the theory of topological semigroups. For example the

well-known O. Andersen's result [1] states that a (0–) simple semigroup is completely (0–) simple if and only if it does not contain the bicyclic semigroup. The bicyclic semigroup does not embed into stable semigroups [15].

**Remark 1.1.** We observe that the bicyclic semigroup is isomorphic to the semigroup  $\mathcal{C}_{\mathbb{N}}(\alpha, \beta)$  which is generated by injective partial transformations  $\alpha$  and  $\beta$  of the set of positive integers  $\mathbb{N}$ , defined as follows:

$$\begin{aligned} (n)\alpha &= n + 1 & \text{if } n \geq 1; \\ (n)\beta &= n - 1 & \text{if } n > 1 \end{aligned}$$

(see Exercise IV.1.11(ii) in [20]).

Recall from [11] that a *partially-ordered group* is a group  $(G, \cdot)$  equipped with a partial order  $\leq$  that is translation-invariant; in other words,  $\leq$  has the property that, for all  $a, b, g \in G$ , if  $a \leq b$  then  $a \cdot g \leq b \cdot g$  and  $g \cdot a \leq g \cdot b$ .

Later by  $e$  we denote the identity of a group  $G$ . The set  $G^+ = \{x \in G \mid e \leq x\}$  in a partially ordered group  $G$  is called the *positive cone*, or the *integral part*, of  $G$  and satisfies the properties:

- 1)  $G^+ \cdot G^+ \subseteq G^+$ ;
- 2)  $G^+ \cap (G^+)^{-1} = \{e\}$ ; and
- 3)  $x^{-1} \cdot G^+ \cdot x \subseteq G^+$  for all  $x \in G$ .

Any subset  $P$  of a group  $G$  that satisfies the conditions 1)–3) induces a partial order on  $G$  ( $x \leq y$  if and only if  $x^{-1} \cdot y \in P$ ) for which  $P$  is the positive cone.

A *linearly ordered* or *totally ordered group* is an ordered group  $G$  such that the order relation “ $\leq$ ” is total [7].

In the remainder we shall assume that  $G$  is a linearly ordered group.

For every  $g \in G$  we denote

$$G^+(g) = \{x \in G \mid g \leq x\}.$$

The set  $G^+(g)$  is called a *positive cone on element  $g$*  in  $G$ .

For arbitrary elements  $g, h \in G$  we consider a partial map  $\alpha_h^g: G \rightarrow G$  defined by the formula

$$(x)\alpha_h^g = x \cdot g^{-1} \cdot h, \quad \text{for } x \in G^+(g).$$

We observe that Lemma XIII.1 from [7] implies that for such partial map  $\alpha_h^g: G \rightarrow G$  the restriction  $\alpha_h^g: G^+(g) \rightarrow G^+(h)$  is a bijective map.

We denote

$$\mathcal{B}(G) = \{\alpha_h^g: G \rightarrow G \mid g, h \in G\} \text{ and } \mathcal{B}^+(G) = \{\alpha_h^g: G \rightarrow G \mid g, h \in G^+\},$$

and consider on the sets  $\mathcal{B}(G)$  and  $\mathcal{B}^+(G)$  the operation of the composition of partial maps. Simple verifications show that

$$\alpha_h^g \cdot \alpha_l^k = \alpha_b^a, \quad \text{where } a = (h \vee k) \cdot h^{-1} \cdot g \quad \text{and} \quad b = (h \vee k) \cdot k^{-1} \cdot l, \quad (1)$$

for  $g, h, k, l \in G$ . Therefore, property 1) of the positive cone and condition (1) imply that  $\mathcal{B}(G)$  and  $\mathcal{B}^+(G)$  are subsemigroups of  $\mathcal{I}_G$ .

**Proposition 1.2.** *Let  $G$  be a linearly ordered group. Then the following assertions hold:*

- (i) *elements  $\alpha_h^g$  and  $\alpha_g^h$  are inverses of each other in  $\mathcal{B}(G)$  for all  $g, h \in G$  (resp.,  $\mathcal{B}^+(G)$  for all  $g, h \in G^+$ );*
- (ii) *an element  $\alpha_h^g$  of the semigroup  $\mathcal{B}(G)$  (resp.,  $\mathcal{B}^+(G)$ ) is an idempotent if and only if  $g = h$ ;*
- (iii)  *$\mathcal{B}(G)$  and  $\mathcal{B}^+(G)$  are inverse subsemigroups of  $\mathcal{I}_G$ ;*
- (iv) *the semigroup  $\mathcal{B}(G)$  (resp.,  $\mathcal{B}^+(G)$ ) is isomorphic to  $S_G = G \times G$  (resp.,  $S_G^+ = G^+ \times G^+$ ) with the semigroup operation:*

$$(a, b) \cdot (c, d) = \begin{cases} (c \cdot b^{-1} \cdot a, d), & \text{if } b < c; \\ (a, d), & \text{if } b = c; \\ (a, b \cdot c^{-1} \cdot d), & \text{if } b > c, \end{cases}$$

where  $a, b, c, d \in G$  (resp.,  $a, b, c, d \in G^+$ ).

*Proof.* (i) Condition (1) implies that

$$\alpha_h^g \cdot \alpha_g^h \cdot \alpha_h^g = \alpha_h^g \quad \text{and} \quad \alpha_g^h \cdot \alpha_h^g \cdot \alpha_g^h = \alpha_g^h,$$

and hence  $\alpha_h^g$  and  $\alpha_g^h$  are inverse elements in  $\mathcal{B}(G)$  (resp.,  $\mathcal{B}^+(G)$ ).

Statement (ii) follows from the property of the semigroup  $\mathcal{I}_G$  that  $\alpha \in \mathcal{I}_G$  is an idempotent if and only if  $\alpha: \text{dom } \alpha \rightarrow \text{ran } \alpha$  is an identity map.

Statements (i), (ii) and Theorem 1.17 from [8] imply statement (iii).

Statement (iv) is a corollary of condition (1).  $\square$

**Remark 1.3.** We observe that Proposition 1.2 implies that:

- (1) if  $G$  is the additive group of integers  $(\mathbb{Z}, +)$  with usual linear order  $\leq$  then the semigroup  $\mathcal{B}^+(G)$  is isomorphic to the bicyclic semigroup  $\mathcal{C}(p, q)$ ;
- (2) if  $G$  is the additive group of real numbers  $(\mathbb{R}, +)$  with usual linear order  $\leq$  then the semigroup  $\mathcal{B}(G)$  is isomorphic to  $B_{(-\infty, \infty)}^2$  (see [17, 18]) and the semigroup  $\mathcal{B}^+(G)$  is isomorphic to  $B_{[0, \infty)}^1$  (see [2, 3, 4, 5, 6]) and
- (3) the semigroup  $\mathcal{B}^+(G)$  is isomorphic to the semigroup  $S(G)$  which is defined in [9, 10].

We shall say that a linearly ordered group  $G$  is a *d-group* if for every element  $g \in G^+ \setminus \{e\}$  there exists  $x \in G^+ \setminus \{e\}$  such that  $x < g$ . We observe that a linearly ordered group  $G$  is a *d-group* if and only if the set  $G^+ \setminus \{e\}$  does not contain a minimal element.

**Definition 1.4.** Suppose that  $G$  is a linearly ordered  $d$ -group. For every  $g \in G$  we denote

$$\mathring{G}^+(g) = \{x \in G \mid g < x\}.$$

The set  $\mathring{G}^+(g)$  is called a *o-positive cone on element  $g$*  in  $G$ .

For arbitrary elements  $g, h \in G$  we consider a partial map  $\mathring{\alpha}_h^g: G \rightarrow G$  defined by the formula

$$(x)\mathring{\alpha}_h^g = x \cdot g^{-1} \cdot h, \quad \text{for } x \in \mathring{G}^+(g).$$

We observe that Lemma XIII.1 from [7] implies that for such partial map  $\mathring{\alpha}_h^g: G \rightarrow G$  the restriction  $\mathring{\alpha}_h^g: \mathring{G}^+(g) \rightarrow \mathring{G}^+(h)$  is a bijective map.

We denote

$$\mathring{\mathcal{B}}(G) = \{\mathring{\alpha}_h^g: G \rightarrow G \mid g, h \in G\} \quad \text{and} \quad \mathring{\mathcal{B}}^+(G) = \{\mathring{\alpha}_h^g: G \rightarrow G \mid g, h \in G^+\},$$

and consider on the sets  $\mathring{\mathcal{B}}(G)$  and  $\mathring{\mathcal{B}}^+(G)$  the operation of the composition of partial maps. Simple verifications show that

$$\mathring{\alpha}_h^g \cdot \mathring{\alpha}_l^k = \mathring{\alpha}_b^a, \quad \text{where } a = (h \vee k) \cdot h^{-1} \cdot g \quad \text{and} \quad b = (h \vee k) \cdot k^{-1} \cdot l, \quad (2)$$

for  $g, h, k, l \in G$ . Therefore, property 1) of the positive cone and condition (2) imply that  $\mathring{\mathcal{B}}(G)$  and  $\mathring{\mathcal{B}}^+(G)$  are subsemigroups of the symmetric inverse semigroup  $\mathcal{I}_G$ .

**Proposition 1.5.** *If  $G$  is a linearly ordered  $d$ -group then the semigroups  $\mathring{\mathcal{B}}(G)$  and  $\mathring{\mathcal{B}}^+(G)$  are isomorphic to  $\mathcal{B}(G)$  and  $\mathcal{B}^+(G)$ , respectively.*

*Proof.* A map  $\mathfrak{h}: \mathcal{B}(G) \rightarrow \mathring{\mathcal{B}}(G)$  (resp.,  $\mathfrak{h}: \mathcal{B}^+(G) \rightarrow \mathring{\mathcal{B}}^+(G)$ ) we define by the formula:

$$(\alpha_h^g)\mathfrak{h} = \mathring{\alpha}_h^g, \quad \text{for } g, h \in G \text{ (resp., } g, h \in G^+).$$

Simple verifications show that such map  $\mathfrak{h}$  is an isomorphism of the semigroups  $\mathring{\mathcal{B}}(G)$  and  $\mathcal{B}(G)$  (resp.,  $\mathring{\mathcal{B}}^+(G)$  and  $\mathcal{B}^+(G)$ ).  $\square$

Suppose that  $G$  is a linearly ordered  $d$ -group. Then obviously  $\mathring{\mathcal{B}}(G) \cap \mathcal{B}(G) = \emptyset$  and  $\mathring{\mathcal{B}}^+(G) \cap \mathcal{B}^+(G) = \emptyset$ . We define

$$\overline{\mathcal{B}}(G) = \mathring{\mathcal{B}}(G) \cup \mathcal{B}(G) \quad \text{and} \quad \overline{\mathcal{B}}^+(G) = \mathring{\mathcal{B}}^+(G) \cup \mathcal{B}^+(G).$$

**Proposition 1.6.** *If  $G$  is a linearly ordered  $d$ -group then  $\overline{\mathcal{B}}(G)$  and  $\overline{\mathcal{B}}^+(G)$  are inverse semigroups.*

*Proof.* Since  $\mathring{\mathcal{B}}(G)$ ,  $\mathcal{B}(G)$ ,  $\mathring{\mathcal{B}}^+(G)$  and  $\mathcal{B}^+(G)$  are inverse subsemigroups of the symmetric inverse semigroup  $\mathcal{I}_G$  over the group  $G$  we conclude that it is sufficient to show that  $\overline{\mathcal{B}}(G)$  and  $\overline{\mathcal{B}}^+(G)$  are subsemigroups of  $\mathcal{I}_G$ .

We fix arbitrary elements  $g, h, k, l \in G$ . Since  $\alpha_h^g, \alpha_l^k, \dot{\alpha}_h^g$  and  $\dot{\alpha}_l^k$  are partial injective maps from  $G$  into  $G$  we have that

$$\alpha_h^g \cdot \dot{\alpha}_l^k = \begin{cases} \dot{\alpha}_l^{k \cdot h^{-1} \cdot g}, & \text{if } h < k; \\ \dot{\alpha}_l^g, & \text{if } h = k; \\ \alpha_{h \cdot k^{-1} \cdot l}^g, & \text{if } h > k \end{cases} \quad \text{and} \quad \dot{\alpha}_h^g \cdot \alpha_l^k = \begin{cases} \alpha_l^{k \cdot h^{-1} \cdot g}, & \text{if } h < k; \\ \dot{\alpha}_l^g, & \text{if } h = k; \\ \dot{\alpha}_{h \cdot k^{-1} \cdot l}^g, & \text{if } h > k. \end{cases}$$

Hence  $\overline{\mathcal{B}}(G)$  is a subsemigroup of  $\mathcal{I}_G$ .

Similar arguments and property 1) of the positive cone imply that  $\overline{\mathcal{B}}^+(G)$  is a subsemigroup of  $\mathcal{I}_G$ . This completes the proof of our proposition.  $\square$

In our paper we study semigroups  $\mathcal{B}(G)$  and  $\mathcal{B}^+(G)$  for a linearly ordered group  $G$ , and semigroups  $\overline{\mathcal{B}}(G)$  and  $\overline{\mathcal{B}}^+(G)$  for a linearly ordered  $d$ -group  $G$ . We describe Green's relations on the semigroups  $\mathcal{B}(G)$ ,  $\mathcal{B}^+(G)$ ,  $\overline{\mathcal{B}}(G)$  and  $\overline{\mathcal{B}}^+(G)$ , their bands and show that they are simple, and moreover the semigroups  $\mathcal{B}(G)$  and  $\mathcal{B}^+(G)$  are bisimple. We show that for a commutative linearly ordered group  $G$  all non-trivial congruences on the semigroup  $\mathcal{B}(G)$  (and  $\mathcal{B}^+(G)$ ) are group congruences if and only if the group  $G$  is archimedean. Also, we describe the structure of group congruences on the semigroups  $\mathcal{B}(G)$ ,  $\mathcal{B}^+(G)$ ,  $\overline{\mathcal{B}}(G)$  and  $\overline{\mathcal{B}}^+(G)$ .

## 2. ALGEBRAIC PROPERTIES OF THE SEMIGROUPS $\mathcal{B}(G)$ AND $\mathcal{B}^+(G)$

**Proposition 2.1.** *Let  $G$  be a linearly ordered group. Then the following assertions hold:*

- (i) *if  $\alpha_g^g, \alpha_h^h \in E(\mathcal{B}(G))$  (resp.,  $\alpha_g^g, \alpha_h^h \in E(\mathcal{B}^+(G))$ ), then  $\alpha_g^g \preceq \alpha_h^h$  if and only if  $g \geq h$  in  $G$  (resp., in  $G^+$ );*
- (ii) *the semilattice  $E(\mathcal{B}(G))$  (resp.,  $E(\mathcal{B}^+(G))$ ) is isomorphic to  $G$  (resp.,  $G^+$ ), considered as a  $\vee$ -semilattice under the mapping  $(\alpha_g^g)\mathbf{i} = g$ ;*
- (iii)  *$\alpha_h^g \mathcal{R} \alpha_l^k$  in  $\mathcal{B}(G)$  (resp., in  $\mathcal{B}^+(G)$ ) if and only if  $g = k$  in  $G$  (resp., in  $G^+$ );*
- (iv)  *$\alpha_h^g \mathcal{L} \alpha_l^k$  in  $\mathcal{B}(G)$  (resp., in  $\mathcal{B}^+(G)$ ) if and only if  $h = l$  in  $G$  (resp., in  $G^+$ );*
- (v)  *$\alpha_h^g \mathcal{H} \alpha_l^k$  in  $\mathcal{B}(G)$  (resp., in  $\mathcal{B}^+(G)$ ) if and only if  $g = k$  and  $h = l$  in  $G$  (resp., in  $G^+$ ), and hence every  $\mathcal{H}$ -class in  $\mathcal{B}(G)$  (resp., in  $\mathcal{B}^+(G)$ ) is a singleton set;*
- (vi)  *$\alpha_h^g \mathcal{D} \alpha_l^k$  in  $\mathcal{B}(G)$  (resp., in  $\mathcal{B}^+(G)$ ) for all  $g, h, k, l \in G$  and hence  $\mathcal{B}(G)$  (resp.,  $\mathcal{B}^+(G)$ ) is a bisimple semigroup;*
- (vii)  *$\mathcal{B}(G)$  (resp.,  $\mathcal{B}^+(G)$ ) is a simple semigroup.*

*Proof.* Statements (i) and (ii) are trivial and follow from the definition of the semigroup  $\mathcal{B}(G)$ .

(iii) Let  $\alpha_h^g, \alpha_l^k \in \mathcal{B}(G)$  be such that  $\alpha_h^g \mathcal{R} \alpha_l^k$ . Since  $\alpha_h^g \mathcal{B}(G) = \alpha_l^k \mathcal{B}(G)$  and  $\mathcal{B}(G)$  is an inverse semigroup, Theorem 1.17 from [8] implies that

$\alpha_h^g \mathcal{B}(G) = \alpha_h^g (\alpha_h^g)^{-1} \mathcal{B}(G)$  and  $\alpha_l^k \mathcal{B}(G) = \alpha_l^k (\alpha_l^k)^{-1} \mathcal{B}(G)$ , and hence  $\alpha_g^g = \alpha_h^g (\alpha_h^g)^{-1} = \alpha_l^k (\alpha_l^k)^{-1} = \alpha_k^k$ . Therefore we get that  $g = k$ .

Conversely, let  $\alpha_h^g, \alpha_l^k \in \mathcal{B}(G)$  be such that  $g = k$ . Then  $\alpha_h^g (\alpha_h^g)^{-1} = \alpha_l^k (\alpha_l^k)^{-1}$ . Since  $\mathcal{B}(G)$  is an inverse semigroup, Theorem 1.17 from [8] implies that  $\alpha_h^g \mathcal{B}(G) = \alpha_h^g (\alpha_h^g)^{-1} \mathcal{B}(G) = \alpha_l^k \mathcal{B}(G)$  and hence  $\alpha_h^g \mathcal{R} \alpha_l^k$  in  $\mathcal{B}(G)$ .

The proof of statement (iv) is similar to (iii).

Statement (v) follows from statements (iii) and (iv).

(vi) For every  $g, h \in \mathcal{B}(G)$  we have that  $\alpha_h^g (\alpha_h^g)^{-1} = \alpha_g^g$  and  $(\alpha_h^g)^{-1} \alpha_h^g = \alpha_h^h$ , and hence by statement (ii), Proposition 1.2 and Lemma 1.1 from [19] we get that  $\mathcal{B}(G)$  is a bisimple semigroup.

(vii) Since every two  $\mathcal{D}$ -equivalent elements of an arbitrary semigroup  $S$  are  $\mathcal{J}$ -equivalent (see [8, Section 2.1]) we have that  $\mathcal{B}(G)$  is a simple semigroup.

The proof of the proposition for the semigroup  $\mathcal{B}^+(G)$  is similar.  $\square$

Given two partially ordered sets  $(A, \leq_A)$  and  $(B, \leq_B)$ , the *lexicographical order*  $\leq_{\text{lex}}$  on the Cartesian product  $A \times B$  is defined as follows:

$$(a, b) \leq_{\text{lex}} (a', b') \quad \text{if and only if} \quad a <_A a' \quad \text{or} \quad (a = a' \text{ and } b \leq_B b').$$

In this case we shall say that the partially ordered set  $(A \times B, \leq_{\text{lex}})$  is the *lexicographic product* of partially ordered sets  $(A, \leq_A)$  and  $(B, \leq_B)$  and it is denoted by  $A \times_{\text{lex}} B$ . We observe that the lexicographic product of two linearly ordered sets is a linearly ordered set.

**Proposition 2.2.** *Let  $G$  be a linearly ordered  $d$ -group. Then the following assertions hold:*

- (i)  $E(\overline{\mathcal{B}}(G)) = E(\mathcal{B}(G)) \cup E(\overset{\circ}{\mathcal{B}}(G))$  and  $E(\overline{\mathcal{B}}^+(G)) = E(\mathcal{B}^+(G)) \cup E(\overset{\circ}{\mathcal{B}}^+(G))$ .
- (ii) If  $\alpha_g^g, \overset{\circ}{\alpha}_g^g, \alpha_h^h, \overset{\circ}{\alpha}_h^h \in E(\overline{\mathcal{B}}(G))$  (resp.,  $\alpha_g^g, \overset{\circ}{\alpha}_g^g, \alpha_h^h, \overset{\circ}{\alpha}_h^h \in E(\overline{\mathcal{B}}^+(G))$ ) then:
  - (a)  $\alpha_g^g \preceq \alpha_h^h$  if and only if  $g \geq h$  in  $G$  (resp., in  $G^+$ );
  - (b)  $\overset{\circ}{\alpha}_g^g \preceq \overset{\circ}{\alpha}_h^h$  if and only if  $g \geq h$  in  $G$  (resp., in  $G^+$ );
  - (c)  $\alpha_g^g \preceq \overset{\circ}{\alpha}_h^h$  if and only if  $g > h$  in  $G$  (resp., in  $G^+$ );
  - (d)  $\overset{\circ}{\alpha}_g^g \preceq \alpha_h^h$  if and only if  $g \geq h$  in  $G$  (resp., in  $G^+$ ).
- (iii) The semilattice  $E(\overline{\mathcal{B}}(G))$  (resp.,  $E(\overline{\mathcal{B}}^+(G))$ ) is isomorphic to the lexicographic product  $G \times_{\text{lex}} \{0, 1\}$  (resp.,  $G^+ \times_{\text{lex}} \{0, 1\}$ ) of semilattices  $(G, \vee)$  (resp.,  $(G^+, \vee)$ ) and  $(\{0, 1\}, \min)$  under the mapping  $(\alpha_g^g)\mathbf{i} = (g, 1)$  and  $(\overset{\circ}{\alpha}_g^g)\mathbf{i} = (g, 0)$ , and hence  $E(\overline{\mathcal{B}}(G))$  (resp.,  $E(\overline{\mathcal{B}}^+(G))$ ) is a linearly ordered semilattice.
- (iv) The elements  $\alpha$  and  $\beta$  of the semigroup  $\overline{\mathcal{B}}(G)$  (resp.,  $\overline{\mathcal{B}}^+(G)$ ) are  $\mathcal{R}$ -equivalent in  $\overline{\mathcal{B}}(G)$  (resp., in  $\overline{\mathcal{B}}^+(G)$ ) provides either  $\alpha, \beta \in \mathcal{B}(G)$

- (resp.,  $\alpha, \beta \in \overline{\mathcal{B}}^+(G)$ ) or  $\alpha, \beta \in \mathring{\mathcal{B}}(G)$  (resp.,  $\alpha, \beta \in \mathring{\mathcal{B}}^+(G)$ ); and moreover, we have that
- (a)  $\alpha_h^g \mathcal{B} \alpha_l^k$  in  $\overline{\mathcal{B}}(G)$  (resp., in  $\overline{\mathcal{B}}^+(G)$ ) if and only if  $g = k$ ; and
  - (b)  $\mathring{\alpha}_h^g \mathcal{B} \mathring{\alpha}_l^k$  in  $\overline{\mathcal{B}}(G)$  (resp., in  $\overline{\mathcal{B}}^+(G)$ ) if and only if  $g = k$ .
- (v) The elements  $\alpha$  and  $\beta$  of the semigroup  $\overline{\mathcal{B}}(G)$  (resp.,  $\overline{\mathcal{B}}^+(G)$ ) are  $\mathcal{L}$ -equivalent in  $\overline{\mathcal{B}}(G)$  (resp., in  $\overline{\mathcal{B}}^+(G)$ ) provides either  $\alpha, \beta \in \mathcal{B}(G)$  (resp.,  $\alpha, \beta \in \overline{\mathcal{B}}^+(G)$ ) or  $\alpha, \beta \in \mathring{\mathcal{B}}(G)$  (resp.,  $\alpha, \beta \in \mathring{\mathcal{B}}^+(G)$ ); and moreover, we have that
- (a)  $\alpha_h^g \mathcal{L} \alpha_l^k$  in  $\overline{\mathcal{B}}(G)$  (resp., in  $\overline{\mathcal{B}}^+(G)$ ) if and only if  $h = l$ ; and
  - (b)  $\mathring{\alpha}_h^g \mathcal{L} \mathring{\alpha}_l^k$  in  $\overline{\mathcal{B}}(G)$  (resp., in  $\overline{\mathcal{B}}^+(G)$ ) if and only if  $h = l$ .
- (vi) The elements  $\alpha$  and  $\beta$  of the semigroup  $\overline{\mathcal{B}}(G)$  (resp.,  $\overline{\mathcal{B}}^+(G)$ ) are  $\mathcal{H}$ -equivalent in  $\overline{\mathcal{B}}(G)$  (resp., in  $\overline{\mathcal{B}}^+(G)$ ) provides either  $\alpha, \beta \in \mathcal{B}(G)$  (resp.,  $\alpha, \beta \in \overline{\mathcal{B}}^+(G)$ ) or  $\alpha, \beta \in \mathring{\mathcal{B}}(G)$  (resp.,  $\alpha, \beta \in \mathring{\mathcal{B}}^+(G)$ ); and moreover, we have that
- (a)  $\alpha_h^g \mathcal{H} \alpha_l^k$  in  $\overline{\mathcal{B}}(G)$  (resp., in  $\overline{\mathcal{B}}^+(G)$ ) if and only if  $g = k$  and  $h = l$ ;
  - (b)  $\mathring{\alpha}_h^g \mathcal{H} \mathring{\alpha}_l^k$  in  $\overline{\mathcal{B}}(G)$  (resp., in  $\overline{\mathcal{B}}^+(G)$ ) if and only if  $g = k$  and  $h = l$ ; and
  - (c) every  $\mathcal{H}$ -class in  $\overline{\mathcal{B}}(G)$  (resp., in  $\overline{\mathcal{B}}^+(G)$ ) is a singleton set.
- (vii)  $\overline{\mathcal{B}}(G)$  (resp.,  $\overline{\mathcal{B}}^+(G)$ ) is a simple semigroup.
- (viii) The semigroup  $\overline{\mathcal{B}}(G)$  (resp.,  $\overline{\mathcal{B}}^+(G)$ ) has only two distinct  $\mathcal{D}$ -classes which are inverse subsemigroups  $\mathcal{B}(G)$  and  $\mathring{\mathcal{B}}(G)$  (resp.,  $\mathcal{B}^+(G)$  and  $\mathring{\mathcal{B}}^+(G)$ ).

*Proof.* Statements (i), (ii) and (iii) follow from the definition of the semigroup  $\overline{\mathcal{B}}(G)$  and Proposition 1.6.

Proofs of statements (iv), (v) and (vi) follow from Proposition 1.6 and Theorem 1.17 [8] and are similar to statements (ii), (iv) and (v) of Proposition 2.1.

(vii) We shall show that  $\overline{\mathcal{B}}(G) \cdot \alpha \cdot \overline{\mathcal{B}}(G) = \overline{\mathcal{B}}(G)$  for every  $\alpha \in \overline{\mathcal{B}}(G)$ . We fix arbitrary  $\alpha, \beta \in \overline{\mathcal{B}}(G)$  and show that there exist  $\gamma, \delta \in \overline{\mathcal{B}}(G)$  such that  $\gamma \cdot \alpha \cdot \delta = \beta$ .

We consider the following cases:

- (1)  $\alpha = \alpha_h^g \in \mathcal{B}(G)$  and  $\beta = \alpha_l^k \in \mathcal{B}(G)$ ;
- (2)  $\alpha = \alpha_h^g \in \mathcal{B}(G)$  and  $\beta = \mathring{\alpha}_l^k \in \mathring{\mathcal{B}}(G)$ ;
- (3)  $\alpha = \mathring{\alpha}_h^g \in \mathring{\mathcal{B}}(G)$  and  $\beta = \alpha_l^k \in \mathcal{B}(G)$ ;
- (4)  $\alpha = \mathring{\alpha}_h^g \in \mathring{\mathcal{B}}(G)$  and  $\beta = \mathring{\alpha}_l^k \in \mathring{\mathcal{B}}(G)$ ,

where  $g, h, k, l \in G$ .

We put:



$$\begin{aligned}
&\gamma = \alpha_g^k \text{ and } \delta = \alpha_l^h \text{ in case (1);} \\
&\gamma = \alpha_g^k \text{ and } \delta = \alpha_l^h \text{ in case (2);} \\
&\gamma = \alpha_a^k \text{ and } \delta = \alpha_l^{a \cdot g^{-1} \cdot h}, \text{ where } a \in G^+(g) \setminus \{g\}, \text{ in case (3);} \\
&\gamma = \alpha_g^k \text{ and } \delta = \alpha_l^h \text{ in case (4).}
\end{aligned}$$

Elementary verifications show that  $\gamma \cdot \alpha \cdot \delta = \beta$ , and this completes the proof of assertion (vii).

Statement (viii) follows from statements (iv) and (v).

The proof of the statements of the proposition for the semigroup  $\overline{\mathcal{B}}^+(G)$  is similar.  $\square$

**Proposition 2.3.** *Let  $G$  be a linearly ordered group. Then for any distinct elements  $g$  and  $h$  in  $G$  such that  $g \leq h$  in  $G$  (resp., in  $G^+$ ) the subsemigroup  $\mathcal{C}(g, h)$  of  $\mathcal{B}(G)$  (resp.,  $\mathcal{B}^+(G)$ ), which is generated by elements  $\alpha_g^g$  and  $\alpha_h^h$ , is isomorphic to the bicyclic semigroup, and hence for every idempotent  $\alpha_g^g$  in  $\mathcal{B}(G)$  (resp., in  $\mathcal{B}^+(G)$ ) there exists a subsemigroup  $\mathcal{C}$  in  $\mathcal{B}(G)$  (resp., in  $\mathcal{B}^+(G)$ ) such that  $\alpha_g^g$  is a unit of  $\mathcal{C}$  and  $\mathcal{C}$  is isomorphic to the bicyclic semigroup.*

*Proof.* Since the semigroup  $\mathcal{C}$  which is generated by elements  $\alpha_h^g$  and  $\alpha_g^h$  is isomorphic to the semigroup  $\mathcal{C}_{\mathbb{N}}(\alpha, \beta)$  (this isomorphism  $i: \mathcal{C} \rightarrow \mathcal{C}_{\mathbb{N}}(\alpha, \beta)$  we can determine on generating elements of  $\mathcal{C}$  by the formulae  $(\alpha_h^g)i = \alpha$  and  $(\alpha_g^h)i = \beta$ ) we conclude that the first part of the proposition follows from Remark 1.1. Obviously, the element  $\alpha_g^g$  is a unity of the semigroup  $\mathcal{C}$ .  $\square$

### 3. CONGRUENCES ON THE SEMIGROUPS $\mathcal{B}(G)$ AND $\mathcal{B}^+(G)$

The following lemma follows from the definition of a congruence on a semilattice:

**Lemma 3.1.** *Let  $\mathfrak{C}$  be an arbitrary congruence on a semilattice  $S$  and let  $\preccurlyeq$  be the natural partial order on  $S$ . Let  $a$  and  $b$  be idempotents of the semigroup  $S$  such that  $a\mathfrak{C}b$ . Then the relation  $a \preccurlyeq b$  implies that  $a\mathfrak{C}c$  for all idempotents  $c \in S$  such that  $a \preccurlyeq c \preccurlyeq b$ .*

A linearly ordered group  $G$  is called *archimedean* if for each  $a, b \in G^+ \setminus \{e\}$  there exist positive integers  $m$  and  $n$  such that  $b \leq a^m$  and  $a \leq b^n$  [7]. Linearly ordered archimedean groups may be described as follows (**Hölder's Theorem**): *A linearly ordered group is Archimedean if and only if it is isomorphic to some subgroup of the additive group of real numbers with the natural order* [13].

**Theorem 3.2.** *Let  $G$  be an archimedean linearly ordered group. Then every non-trivial congruence on  $\mathcal{B}^+(G)$  is a group congruence.*

*Proof.* Suppose that  $\mathfrak{C}$  is a non-trivial congruence on the semigroup  $\mathcal{B}^+(G)$ . Then there exist distinct elements  $\alpha_b^a$  and  $\alpha_d^c$  of the semigroup  $\mathcal{B}^+(G)$  such

that  $\alpha_b^a \mathfrak{C} \alpha_d^c$ . Since by Proposition 2.1(v) every  $\mathcal{H}$ -class of the semigroup  $\mathcal{B}^+(G)$  is a singleton set we conclude that either  $\alpha_b^a \cdot (\alpha_b^a)^{-1} \neq \alpha_d^c \cdot (\alpha_d^c)^{-1}$  or  $(\alpha_b^a)^{-1} \cdot \alpha_b^a \neq (\alpha_d^c)^{-1} \cdot \alpha_d^c$ . We shall consider case  $\alpha_a^a = \alpha_b^a \cdot (\alpha_b^a)^{-1} \neq \alpha_d^c \cdot (\alpha_d^c)^{-1} = \alpha_c^c$ . In the other case the proof is similar. Since by Proposition 2.1(ii) the band  $E(\mathcal{B}^+(G))$  is a linearly ordered semilattice without loss of generality we can assume that  $\alpha_c^c \preccurlyeq \alpha_a^a$ . Then by Proposition 2.1(i) we have that  $a \leq c$  in  $G$ . Since  $\alpha_b^a \mathfrak{C} \alpha_d^c$  and  $\mathcal{B}^+(G)$  is an inverse semigroup Lemma III.1.1 from [20] implies that  $(\alpha_b^a \cdot (\alpha_b^a)^{-1}) \mathfrak{C} (\alpha_d^c \cdot (\alpha_d^c)^{-1})$ , i.e.,  $\alpha_a^a \mathfrak{C} \alpha_c^c$ . Then we have that

$$\begin{aligned} \alpha_a^c \cdot \alpha_a^a \cdot \alpha_c^a &= \alpha_c^c; \\ \alpha_a^c \cdot \alpha_c^c \cdot \alpha_c^a &= \alpha_{c \cdot a^{-1} \cdot c}^c; \\ \alpha_a^c \cdot \alpha_{c \cdot a^{-1} \cdot c}^c \cdot \alpha_c^a &= \alpha_{c \cdot (a^{-1} \cdot c)^2}^c; \\ &\dots \dots \\ \alpha_a^c \cdot \alpha_{c \cdot (a^{-1} \cdot c)^{n-1}}^c \cdot \alpha_c^a &= \alpha_{c \cdot (a^{-1} \cdot c)^n}^c. \end{aligned}$$

and hence  $\alpha_a^a \mathfrak{C} \alpha_{c \cdot (a^{-1} \cdot c)^n}^c$  for every non-negative integer  $n$ . Since  $a < c$  in  $G$  we get that  $a^{-1} \cdot c$  is a positive element of the linearly ordered group  $G$ . Since the linearly ordered group  $G$  is archimedean we conclude that for every  $g \in G$  with  $g > a$  there exists a positive integer  $n$  such that  $a^{-1} \cdot g < (a^{-1} \cdot c)^n$  and hence  $g < c \cdot (a^{-1} \cdot c)^{n-1}$ . Therefore Lemma 3.1 and Proposition 2.1(i) imply that  $\alpha_a^a \mathfrak{C} \alpha_g^g$  for every  $g \in G$  such that  $a \leq g$ .

If  $a = e$  then we have that all idempotents of the semigroup  $\mathcal{B}^+(G)$  are  $\mathfrak{C}$ -equivalent. Since the semigroup  $\mathcal{B}^+(G)$  is inverse we conclude that the quotient semigroup  $\mathcal{B}^+(G)/\mathfrak{C}$  contains only one idempotent and by Lemma II.1.10 from [20] the semigroup  $\mathcal{B}^+(G)/\mathfrak{C}$  is a group.

Suppose that  $e < a$ . Then by Proposition 2.3 we have that the semigroup  $\mathcal{C}^*$  which is generated by elements  $\alpha_g^e$  and  $\alpha_e^g$  is isomorphic to the bicyclic semigroup for every element  $g$  in  $G^+$  such that  $e < a \leq g$ . Hence we have that the following conditions hold :

$$\dots \preccurlyeq \alpha_{g^i}^{g^i} \preccurlyeq \alpha_{g^{i-1}}^{g^{i-1}} \preccurlyeq \dots \preccurlyeq \alpha_g^g \preccurlyeq \alpha_a^a \quad \text{and}$$

$$\alpha_{g^i}^{g^i} \neq \alpha_{g^j}^{g^j} \quad \text{for distinct positive integers } i \text{ and } j,$$

in  $E(\mathcal{B}^+(G))$ . Since the linearly ordered group  $G$  is archimedean we conclude that  $\alpha_a^a \mathfrak{C} \alpha_{g^i}^{g^i}$  for every positive integer  $i$ . Since the semigroup  $\mathcal{C}^*$  is isomorphic to the bicyclic semigroup we have that Corollary 1.32 of [8] and Lemma 3.1 imply that all idempotents of the semigroup  $\mathcal{B}^+(G)$  are  $\mathfrak{C}$ -equivalent. Since the semigroup  $\mathcal{B}^+(G)$  is inverse we conclude that the quotient semigroup  $\mathcal{B}^+(G)/\mathfrak{C}$  contains only one idempotent and by Lemma II.1.10 from [20] the semigroup  $\mathcal{B}^+(G)/\mathfrak{C}$  is a group.  $\square$

**Theorem 3.3.** *Let  $G$  be an archimedean linearly ordered group. Then every non-trivial congruence on  $\mathcal{B}(G)$  is a group congruence.*

*Proof.* Suppose that  $\mathfrak{C}$  is a non-trivial congruence on the semigroup  $\mathcal{B}(G)$ . Similar arguments as in the proof of Theorem 3.2 imply that there exist distinct idempotents  $\alpha_a^a$  and  $\alpha_b^b$  in the semigroup  $\mathcal{B}(G)$  such that  $\alpha_a^a \mathfrak{C} \alpha_b^b$  and  $\alpha_b^b \preccurlyeq \alpha_a^a$ , for  $a, b \in G$  with  $a \leq b$  in  $G$ . Then we have that

$$\alpha_a^e \cdot \alpha_a^a \cdot \alpha_e^a = \alpha_e^e \quad \text{and} \quad \alpha_a^e \cdot \alpha_b^b \cdot \alpha_e^a = \alpha_b^{b \cdot a^{-1}} \cdot \alpha_e^a = \alpha_{b \cdot a^{-1}}^{b \cdot a^{-1}},$$

and hence  $\alpha_e^e \mathfrak{C} \alpha_{b \cdot a^{-1}}^{b \cdot a^{-1}}$ . Since  $a \leq b$  in  $G$  we conclude that  $e \leq b \cdot a^{-1}$  in  $G$  and hence Theorem 3.2 implies that  $\alpha_e^e \mathfrak{C} \alpha_d^d$  for all  $c, d \in G^+$ .

We fix an arbitrary element  $g \in G \setminus G^+$ . Then we have that  $g^{-1} \in G^+ \setminus \{e\}$  and hence  $\alpha_e^e \mathfrak{C} \alpha_{g^{-1}}^{g^{-1}}$ . Since

$$\alpha_e^g \cdot \alpha_e^e \cdot \alpha_g^e = \alpha_g^g \quad \text{and} \quad \alpha_e^g \cdot \alpha_{g^{-1}}^{g^{-1}} \cdot \alpha_g^e = \alpha_{g^{-1}}^{g^{-1} \cdot e \cdot g} \cdot \alpha_g^e = \alpha_{g^{-1}}^e \cdot \alpha_g^e = \alpha_{g^{-1} \cdot e \cdot g}^e = \alpha_e^e$$

we conclude that  $\alpha_e^e \mathfrak{C} \alpha_g^g$ . Therefore all idempotents of the semigroup  $\mathcal{B}(G)$  are  $\mathfrak{C}$ -equivalent. Since the semigroup  $\mathcal{B}(G)$  is inverse we conclude that the quotient semigroup  $\mathcal{B}(G)/\mathfrak{C}$  contains only one idempotent and by Lemma II.1.10 from [20] the semigroup  $\mathcal{B}(G)/\mathfrak{C}$  is a group.  $\square$

**Remark 3.4.** We observe that Proposition 1.5 implies that if  $G$  is a linearly ordered  $d$ -group then the statements similar to Propositions 2.1 and 2.3 and Theorems 3.2 and 3.3 hold for the semigroups  $\mathring{\mathcal{B}}(G)$  and  $\mathring{\mathcal{B}}^+(G)$ .

**Theorem 3.5.** *If  $G$  is the lexicographic product  $A \times_{\text{lex}} H$  of non-singleton linearly ordered groups  $A$  and  $H$ , then the semigroups  $\mathcal{B}(G)$  and  $\mathcal{B}^+(G)$  have non-trivial non-group congruences.*

*Proof.* We define a relation  $\sim_{\mathfrak{C}}$  on the semigroup  $\mathcal{B}(G)$  as follows:

$$\alpha_{(c_1, d_1)}^{(a_1, b_1)} \sim_{\mathfrak{C}} \alpha_{(c_2, d_2)}^{(a_2, b_2)} \quad \text{if and only if} \quad a_1 = a_2, c_1 = c_2 \quad \text{and} \quad d_1^{-1} b_1 = d_2^{-1} b_2.$$

Simple verifications show that  $\sim_{\mathfrak{C}}$  is an equivalence relation on the semigroup  $\mathcal{B}(G)$ .

Next we shall prove that  $\sim_{\mathfrak{c}}$  is a congruence on  $\mathcal{B}(G)$ . Suppose that  $\alpha_{(c_1, d_1)}^{(a_1, b_1)} \sim_{\mathfrak{c}} \alpha_{(c_2, d_2)}^{(a_2, b_2)}$  for some  $\alpha_{(c_1, d_1)}^{(a_1, b_1)}, \alpha_{(c_2, d_2)}^{(a_2, b_2)} \in \mathcal{B}(G)$ . Let  $\alpha_{(x, y)}^{(u, v)}$  be an arbitrary element of  $\mathcal{B}(G)$ . Then we have that

$$\begin{aligned} \alpha_{(m_1, n_1)}^{(k_1, l_1)} &= \alpha_{(c_1, d_1)}^{(a_1, b_1)} \cdot \alpha_{(x, y)}^{(u, v)} = \begin{cases} \alpha_{(x, y)}^{(u, v) \cdot (c_1, d_1)^{-1} \cdot (a_1, b_1)}, & \text{if } (c_1, d_1) \leq (u, v); \\ \alpha_{(c_1, d_1) \cdot (u, v)^{-1} \cdot (x, y)}^{(a_1, b_1)}, & \text{if } (u, v) \leq (c_1, d_1) \end{cases} = \\ &= \begin{cases} \alpha_{(x, y)}^{(uc_1^{-1}a_1, vd_1^{-1}b_1)}, & \text{if } (c_1, d_1) \leq (u, v); \\ \alpha_{(c_1u^{-1}x, d_1v^{-1}y)}^{(a_1, b_1)}, & \text{if } (u, v) \leq (c_1, d_1) \end{cases} = \\ &= \begin{cases} \alpha_{(x, y)}^{(uc_1^{-1}a_1, vd_1^{-1}b_1)}, & \text{if } c_1 < u; \\ \alpha_{(x, y)}^{(a_1, vd_1^{-1}b_1)}, & \text{if } c_1 = u \text{ and } d_1 \leq v; \\ \alpha_{(c_1u^{-1}x, d_1v^{-1}y)}^{(a_1, b_1)}, & \text{if } u < c_1; \\ \alpha_{(x, d_1v^{-1}y)}^{(a_1, b_1)}, & \text{if } u = c_1 \text{ and } v \leq d_1; \end{cases} \end{aligned}$$

and

$$\begin{aligned} \alpha_{(m_2, n_2)}^{(k_2, l_2)} &= \alpha_{(c_2, d_2)}^{(a_2, b_2)} \cdot \alpha_{(x, y)}^{(u, v)} = \begin{cases} \alpha_{(x, y)}^{(u, v) \cdot (c_2, d_2)^{-1} \cdot (a_2, b_2)}, & \text{if } (c_2, d_2) \leq (u, v); \\ \alpha_{(c_2, d_2) \cdot (u, v)^{-1} \cdot (x, y)}^{(a_2, b_2)}, & \text{if } (u, v) \leq (c_2, d_2) \end{cases} = \\ &= \begin{cases} \alpha_{(x, y)}^{(uc_2^{-1}a_2, vd_2^{-1}b_2)}, & \text{if } (c_2, d_2) \leq (u, v); \\ \alpha_{(c_2u^{-1}x, d_2v^{-1}y)}^{(a_2, b_2)}, & \text{if } (u, v) \leq (c_2, d_2), \end{cases} = \\ &= \begin{cases} \alpha_{(x, y)}^{(uc_2^{-1}a_2, vd_2^{-1}b_2)}, & \text{if } c_2 < u; \\ \alpha_{(x, y)}^{(a_2, vd_2^{-1}b_2)}, & \text{if } c_2 = u \text{ and } d_2 \leq v; \\ \alpha_{(c_2u^{-1}x, d_2v^{-1}y)}^{(a_2, b_2)}, & \text{if } u < c_2; \\ \alpha_{(x, d_2v^{-1}y)}^{(a_2, b_2)}, & \text{if } u = c_2 \text{ and } v \leq d_2; \end{cases} \end{aligned}$$

and since  $a_1 = a_2$ ,  $c_1 = c_2$  and  $d_1^{-1}b_1 = d_2^{-1}b_2$  we conclude that the following conditions hold:

- (1) if  $c_1 = c_2 < u$ , then  $k_1 = uc_1^{-1}a_1 = uc_2^{-1}a_2 = k_2$ ,  $m_1 = x = m_2$  and  $n_1^{-1}l_1 = y^{-1}vd_1^{-1}b_1 = y^{-1}vd_2^{-1}b_2 = n_2^{-1}l_2$ ;
- (2) if  $c_1 = c_2 = u$  and  $d_1 \leq v$ , then  $k_1 = a_1 = a_2 = k_2$ ,  $m_1 = x = m_2$  and  $n_1^{-1}l_1 = y^{-1}vd_1^{-1}b_1 = y^{-1}vd_2^{-1}b_2 = n_2^{-1}l_2$ ;
- (3) if  $u < c_1 = c_2$ , then  $k_1 = a_1 = a_2 = k_2$ ,  $m_1 = c_1u^{-1}x = c_2u^{-1}x = m_2$  and  $n_1^{-1}l_1 = y^{-1}vd_1^{-1}b_1 = y^{-1}vd_2^{-1}b_2 = n_2^{-1}l_2$ ;

- (4) if  $u = c_1 = c_2$  and  $v \leq d_1$ , then  $k_1 = a_1 = a_2 = k_2$ ,  $m_1 = x = m_2$  and

$$n_1^{-1}l_1 = y^{-1}vd_1^{-1}b_1 = y^{-1}vd_2^{-1}b_2 = n_2^{-1}l_2.$$

Hence we get that  $\alpha_{(m_1, n_1)}^{(k_1, l_1)} \sim_{\mathfrak{c}} \alpha_{(m_2, n_2)}^{(k_2, l_2)}$ . Similarly we have that

$$\begin{aligned} \alpha_{(r_1, s_1)}^{(p_1, q_1)} &= \alpha_{(x, y)}^{(u, v)} \cdot \alpha_{(c_1, d_1)}^{(a_1, b_1)} = \begin{cases} \alpha_{(c_1, d_1)}^{(a_1, b_1) \cdot (x, y)^{-1} \cdot (u, v)}, & \text{if } (x, y) \leq (a_1, b_1); \\ \alpha_{(x, y) \cdot (a_1, b_1)^{-1} \cdot (c_1, d_1)}^{(u, v)}, & \text{if } (a_1, b_1) \leq (x, y) \end{cases} = \\ &= \begin{cases} \alpha_{(c_1, d_1)}^{(a_1 x^{-1} u, b_1 y^{-1} v)}, & \text{if } (x, y) \leq (a_1, b_1); \\ \alpha_{(x a_1^{-1} c_1, y b_1^{-1} d_1)}^{(u, v)}, & \text{if } (a_1, b_1) \leq (x, y), \end{cases} = \\ &= \begin{cases} \alpha_{(c_1, d_1)}^{(a_1 x^{-1} u, b_1 y^{-1} v)}, & \text{if } x < a_1; \\ \alpha_{(c_1, d_1)}^{(u, b_1 y^{-1} v)}, & \text{if } x = a_1 \text{ and } y \leq b_1; \\ \alpha_{(x a_1^{-1} c_1, y b_1^{-1} d_1)}^{(u, v)}, & \text{if } a_1 < x; \\ \alpha_{(c_1, y b_1^{-1} d_1)}^{(u, v)}, & \text{if } a_1 = x \text{ and } b_1 \leq y, \end{cases} \end{aligned}$$

and

$$\begin{aligned} \alpha_{(r_2, s_2)}^{(p_2, q_2)} &= \alpha_{(x, y)}^{(u, v)} \cdot \alpha_{(c_2, d_2)}^{(a_2, b_2)} = \begin{cases} \alpha_{(c_2, d_2)}^{(a_2, b_2) \cdot (x, y)^{-1} \cdot (u, v)}, & \text{if } (x, y) \leq (a_2, b_2); \\ \alpha_{(x, y) \cdot (a_2, b_2)^{-1} \cdot (c_2, d_2)}^{(u, v)}, & \text{if } (a_2, b_2) \leq (x, y) \end{cases} = \\ &= \begin{cases} \alpha_{(c_2, d_2)}^{(a_2 x^{-1} u, b_2 y^{-1} v)}, & \text{if } (x, y) \leq (a_2, b_2); \\ \alpha_{(x a_2^{-1} c_2, y b_2^{-1} d_2)}^{(u, v)}, & \text{if } (a_2, b_2) \leq (x, y) \end{cases} = \\ &= \begin{cases} \alpha_{(c_2, d_2)}^{(a_2 x^{-1} u, b_2 y^{-1} v)}, & \text{if } x < a_2; \\ \alpha_{(c_2, d_2)}^{(u, b_2 y^{-1} v)}, & \text{if } x = a_2 \text{ and } y \leq b_2; \\ \alpha_{(x a_2^{-1} c_2, y b_2^{-1} d_2)}^{(u, v)}, & \text{if } a_2 < x; \\ \alpha_{(c_2, y b_2^{-1} d_2)}^{(u, v)}, & \text{if } a_2 = x \text{ and } b_2 \leq y, \end{cases} \end{aligned}$$

and since  $a_1 = a_2$ ,  $c_1 = c_2$  and  $d_1^{-1}b_1 = d_2^{-1}b_2$  we conclude that the following conditions hold:

- (1) if  $x < a_1 = a_2$ , then  $p_1 = a_1 x^{-1} u = a_2 x^{-1} u = p_2$ ,  $r_1 = c_1 = c_2 = r_2$  and

$$s_1^{-1}q_1 = d_1^{-1}b_1 y^{-1}v = d_2^{-1}b_2 y^{-1}v = s_2^{-1}q_2;$$

- (2) if  $x = a_1 = a_2$  and  $y \leq b_1$ , then  $p_1 = u = p_2$ ,  $r_1 = c_1 = c_2 = r_2$  and

$$s_1^{-1}q_1 = d_1^{-1}b_1 y^{-1}v = d_2^{-1}b_2 y^{-1}v = s_2^{-1}q_2;$$

- (3) if  $a_1 = a_2 < x$ , then  $p_1 = u = p_2$ ,  $r_1 = x a_1^{-1} c_1 = x a_2^{-1} c_2 = r_2$  and

$$s_1^{-1}q_1 = d_1^{-1}b_1 y^{-1}v = d_2^{-1}b_2 y^{-1}v = s_2^{-1}q_2;$$

(4) if  $a_1 = a_2 = x$  and  $b_1 \leq y$ , then  $p_1 = u = p_2$ ,  $r_1 = c_1 = c_2 = r_2$  and

$$s_1^{-1}q_1 = d_1^{-1}b_1y^{-1}v = d_2^{-1}b_2y^{-1}v = s_2^{-1}q_2.$$

Hence we get that  $\alpha_{(r_1, s_1)}^{(p_1, q_1)} \sim_c \alpha_{(r_2, s_2)}^{(p_2, q_2)}$ .

We fix any  $a_1, a_2, b_1, b_2 \in G$ . If  $a_1 \neq a_2$  then we have that the elements  $\alpha_{(a_1, b_1)}^{(a_1, b_1)}$  and  $\alpha_{(a_2, b_2)}^{(a_2, b_2)}$  are idempotents of the semigroup  $\mathcal{B}(G)$ , and moreover the elements  $\alpha_{(a_1, b_1)}^{(a_1, b_1)}$  and  $\alpha_{(a_2, b_2)}^{(a_2, b_2)}$  are not  $\sim_c$ -equivalent. Since a homomorphic image of an idempotent is an idempotent too, we conclude that  $\left(\alpha_{(a_1, b_1)}^{(a_1, b_1)}\right) \pi_c \neq \left(\alpha_{(a_2, b_2)}^{(a_2, b_2)}\right) \pi_c$ , where  $\pi_c: \mathcal{B}(G) \rightarrow \mathcal{B}(G)/\sim_c$  is the natural homomorphism which is generated by the congruence  $\sim_c$  on the semigroup  $\mathcal{B}(G)$ . This implies that the quotient semigroup  $\mathcal{B}(G)/\sim_c$  is not a group, and hence  $\sim_c$  is not a group congruence on the semigroup  $\mathcal{B}(G)$ .

The proof of the statement that the semigroup  $\mathcal{B}^+(G)$  has a non-trivial non-group congruence is similar.  $\square$

**Theorem 3.6.** *Let  $G$  be a commutative linearly ordered group. Then the following conditions are equivalent:*

- (i)  $G$  is archimedean;
- (ii) every non-trivial congruence on  $\mathcal{B}(G)$  is a group congruence; and
- (iii) every non-trivial congruence on  $\mathcal{B}^+(G)$  is a group congruence.

*Proof.* Implications (i)  $\Rightarrow$  (ii) and (i)  $\Rightarrow$  (iii) follow from Theorems 3.3 and 3.2, respectively.

(ii)  $\Rightarrow$  (i) Suppose the contrary that there exists a non-archimedean commutative linearly ordered group  $G$  such that every non-trivial congruence on  $\mathcal{B}(G)$  is a group congruence. Then by Hahn Theorem (see [12] or [16, Section VII.3, Theorem 1])  $G$  is isomorphic to a lexicographic product  $\prod_{\alpha \in \mathcal{J}} \text{lex} H_\alpha$

of some family of non-singleton subgroups  $\{H_\alpha \mid \alpha \in \mathcal{J}\}$  of the additive group of real numbers with a non-singleton linearly ordered index set  $\mathcal{J}$ . We fix a non-maximal element  $\alpha_0 \in \mathcal{J}$ , and put

$$A = \prod_{\text{lex}} \{H_\alpha \mid \alpha \leq \alpha_0\} \quad \text{and} \quad H = \prod_{\text{lex}} \{H_\alpha \mid \alpha_0 < \alpha\}.$$

Then we have that  $G$  is isomorphic to a lexicographic product  $A \times_{\text{lex}} H$  of non-singleton linearly ordered groups  $A$  and  $H$ , and hence by Theorem 3.5 the semigroup  $\mathcal{B}(G)$  has a non-trivial non-group congruence. The obtained contradiction implies that the group  $G$  is archimedean.

The proof of implication (iii)  $\Rightarrow$  (i) is similar to (ii)  $\Rightarrow$  (i).  $\square$

On the semigroup  $\overline{\mathcal{B}}(G)$  (resp.,  $\overline{\mathcal{B}}^+(G)$ ) we determine a relation  $\sim_{\text{id}}$  in the following way. We define a map  $\text{id}: \overline{\mathcal{B}}(G) \rightarrow \overline{\mathcal{B}}(G)$  (resp.,  $\text{id}: \overline{\mathcal{B}}^+(G) \rightarrow$

$\overline{\mathcal{B}}^+(G)$ ) by the formulae  $(\alpha_h^g)\mathbf{id} = \mathring{\alpha}_h^g$  and  $(\mathring{\alpha}_h^g)\mathbf{id} = \alpha_h^g$  for  $g, h \in G$  (resp.,  $g, h \in G^+$ ). We put

$$\alpha \sim_{\mathbf{id}} \beta \quad \text{if and only if} \quad \alpha = \beta \quad \text{or} \quad (\alpha)\mathbf{id} = \beta \quad \text{or} \quad (\beta)\mathbf{id} = \alpha,$$

for  $\alpha, \beta \in \overline{\mathcal{B}}(G)$  (resp.,  $\alpha, \beta \in \overline{\mathcal{B}}^+(G)$ ). Simple verifications show that  $\sim_{\mathbf{id}}$  is an equivalence relation on the semigroup  $\overline{\mathcal{B}}(G)$  (resp.,  $\overline{\mathcal{B}}^+(G)$ ).

**Proposition 3.7.** *If  $G$  is a linearly ordered  $d$ -group then  $\sim_{\mathbf{id}}$  is a congruence on semigroups  $\overline{\mathcal{B}}(G)$  and  $\overline{\mathcal{B}}^+(G)$ . Moreover, quotient semigroups  $\overline{\mathcal{B}}(G)/\sim_{\mathbf{id}}$  and  $\overline{\mathcal{B}}(G)^+/\sim_{\mathbf{id}}$  are isomorphic to semigroups  $\mathcal{B}(G)$  and  $\mathcal{B}^+(G)$ , respectively.*

*Proof.* It is sufficient to show that if  $\alpha \sim_{\mathbf{id}} \beta$  and  $\gamma \sim_{\mathbf{id}} \delta$  then  $(\alpha \cdot \gamma) \sim_{\mathbf{id}} (\beta \cdot \delta)$  for  $\alpha, \beta, \gamma, \delta \in \overline{\mathcal{B}}(G)$  (resp.,  $\alpha, \beta, \gamma, \delta \in \overline{\mathcal{B}}^+(G)$ ). Since the case  $\alpha = \beta$  and  $\gamma = \delta$  is trivial we consider the following cases:

- (i)  $\alpha = \alpha_b^a, \beta = \mathring{\alpha}_b^a$  and  $\gamma = \delta = \alpha_d^c$ ;
- (ii)  $\alpha = \alpha_b^a, \beta = \mathring{\alpha}_b^a$  and  $\gamma = \delta = \mathring{\alpha}_d^c$ ;
- (iii)  $\alpha = \mathring{\alpha}_b^a, \beta = \alpha_b^a$  and  $\gamma = \delta = \alpha_d^c$ ;
- (iv)  $\alpha = \mathring{\alpha}_b^a, \beta = \alpha_b^a$  and  $\gamma = \delta = \mathring{\alpha}_d^c$ ;
- (v)  $\alpha = \alpha_b^a, \beta = \mathring{\alpha}_b^a, \gamma = \mathring{\alpha}_d^c$  and  $\delta = \alpha_d^c$ ;
- (vi)  $\alpha = \alpha_b^a, \beta = \mathring{\alpha}_b^a, \gamma = \alpha_d^c$  and  $\delta = \mathring{\alpha}_d^c$ ;
- (vii)  $\alpha = \mathring{\alpha}_b^a, \beta = \alpha_b^a, \gamma = \mathring{\alpha}_d^c$  and  $\delta = \alpha_d^c$ ;
- (viii)  $\alpha = \mathring{\alpha}_b^a, \beta = \alpha_b^a, \gamma = \alpha_d^c$  and  $\delta = \mathring{\alpha}_d^c$ ;
- (ix)  $\alpha = \beta = \alpha_b^a, \gamma = \mathring{\alpha}_d^c$  and  $\delta = \alpha_d^c$ ;
- (x)  $\alpha = \beta = \mathring{\alpha}_b^a, \gamma = \mathring{\alpha}_d^c$  and  $\delta = \alpha_d^c$ ;
- (xi)  $\alpha = \beta = \alpha_b^a, \gamma = \alpha_d^c$  and  $\delta = \mathring{\alpha}_d^c$ ; and
- (xii)  $\alpha = \beta = \mathring{\alpha}_b^a, \gamma = \alpha_d^c$  and  $\delta = \mathring{\alpha}_d^c$ ,

where  $a, b, c, d \in G$  (resp.,  $a, b, c, d \in G^+$ ).

In case (i) we have that

$$\alpha \cdot \gamma = \alpha_b^a \cdot \alpha_d^c = \begin{cases} \alpha_d^{c \cdot b^{-1} \cdot a}, & \text{if } b < c; \\ \alpha_d^a, & \text{if } b = c; \\ \alpha_{b \cdot c^{-1} \cdot d}^a, & \text{if } b > c, \end{cases} \quad \text{and} \quad \beta \cdot \delta = \mathring{\alpha}_b^a \cdot \alpha_d^c = \begin{cases} \mathring{\alpha}_d^{c \cdot b^{-1} \cdot a}, & \text{if } b < c; \\ \mathring{\alpha}_d^a, & \text{if } b = c; \\ \alpha_{b \cdot c^{-1} \cdot d}^a, & \text{if } b > c, \end{cases}$$

and hence  $(\alpha \cdot \gamma) \sim_{\mathbf{id}} (\beta \cdot \delta)$  in  $\overline{\mathcal{B}}(G)$  (resp.,  $\overline{\mathcal{B}}^+(G)$ ). In other cases verifications are similar.

Since the restriction  $\Phi_{\sim_{\mathbf{id}}} |_{\mathcal{B}(G)} : \mathcal{B}(G) \rightarrow \mathcal{B}(G)$  of the natural homomorphism  $\Phi_{\sim_{\mathbf{id}}} : \overline{\mathcal{B}}(G) \rightarrow \mathcal{B}(G)$  is a bijective map we conclude that the semigroup  $(\overline{\mathcal{B}}(G))\Phi_{\sim_{\mathbf{id}}}$  is isomorphic to the semigroup  $\mathcal{B}(G)$ . Similar arguments show that the semigroup  $\overline{\mathcal{B}}^+(G)/\sim_{\mathbf{id}}$  is isomorphic to  $\mathcal{B}^+(G)$ .  $\square$

**Theorem 3.8.** *Let  $G$  be an archimedean linearly ordered  $d$ -group. If  $\mathfrak{C}$  is a non-trivial congruence on  $\overline{\mathcal{B}}(G)$  (resp.,  $\overline{\mathcal{B}}^+(G)$ ) then the quotient semigroup*

$\overline{\mathcal{B}}(G)/\mathfrak{C}$  (resp.,  $\overline{\mathcal{B}}^+(G)/\mathfrak{C}$ ) is either a group or  $\overline{\mathcal{B}}(G)/\mathfrak{C}$  (resp.,  $\overline{\mathcal{B}}^+(G)/\mathfrak{C}$ ) is isomorphic to the semigroup  $\mathcal{B}(G)$  (resp.,  $\mathcal{B}^+(G)$ ).

*Proof.* Since the subsemigroup of idempotents of the semigroup  $\overline{\mathcal{B}}(G)$  is linearly ordered we have that similar arguments as in the proof of Theorem 3.2 imply that there exist distinct idempotents  $\varepsilon$  and  $\iota$  of  $\overline{\mathcal{B}}(G)$  such that  $\varepsilon\iota$  and  $\varepsilon \preceq \iota$ . If the set  $(\varepsilon, \iota) = \{v \in E(\overline{\mathcal{B}}(G)) \mid \varepsilon \prec v \prec \iota\}$  is non-empty then Lemma 3.1 and Theorem 3.2 imply that the quotient semigroup  $\overline{\mathcal{B}}(G)/\mathfrak{C}$  is inverse and has only one idempotent, and hence by Lemma II.1.10 from [20] it is a group. Otherwise there exists  $g \in G$  such that  $\iota = \alpha_g^g$  and  $\varepsilon = \mathring{\alpha}_g^g$ . Since  $\alpha_l^k = \alpha_g^k \cdot \alpha_g^g \cdot \alpha_l^g$  and  $\mathring{\alpha}_l^k = \alpha_g^k \cdot \mathring{\alpha}_g^g \cdot \alpha_l^g$  for every  $k, l \in G$  we conclude that the congruence  $\mathfrak{C}$  coincides with the congruence  $\sim_{\text{id}}$  on  $\overline{\mathcal{B}}(G)$ , and hence by Proposition 3.7 the quotient semigroup  $\overline{\mathcal{B}}(G)/\mathfrak{C}$  is isomorphic to the semigroup  $\mathcal{B}(G)$ .

In the case of the semigroup  $\overline{\mathcal{B}}^+(G)$  the proof is similar.  $\square$

**Theorem 3.9.** *Let  $G$  be a commutative linearly ordered  $d$ -group. Then the following conditions are equivalent:*

- (i)  $G$  is archimedean;
- (ii) every non-trivial congruence on  $\mathring{\mathcal{B}}(G)$  is a group congruence; and
- (iii) every non-trivial congruence on  $\mathring{\mathcal{B}}^+(G)$  is a group congruence;
- (iv) the semigroup  $\overline{\mathcal{B}}(G)$  has a unique non-trivial non-group congruence;
- (v) the semigroup  $\overline{\mathcal{B}}^+(G)$  has a unique non-trivial non-group congruence.

*Proof.* The equivalence of statements (i), (ii) and (iii) follows from Proposition 1.5 and Theorem 3.6. Also Theorem 3.8 implies that implications (i)  $\Rightarrow$  (iv) and (i)  $\Rightarrow$  (v) hold.

Next we shall show that implication (iv)  $\Rightarrow$  (i) holds. Suppose the contrary: there exists a commutative linearly ordered non-archimedean  $d$ -group  $G$  such that the semigroup  $\overline{\mathcal{B}}(G)$  has a unique non-trivial non-group congruence. Then by Proposition 3.7 we have that  $\sim_{\text{id}}$  is a unique non-trivial non-group congruence on the semigroup  $\overline{\mathcal{B}}(G)$ . Therefore, similarly as in the proof of Theorem 3.6 we get that  $G$  is isomorphic to the lexicographic product  $A \times_{\text{lex}} H$  of non-singleton linearly ordered groups  $A$  and  $H$ , and hence by Theorem 3.5 we have that the semigroup  $\mathcal{B}(G)$  has a non-trivial non-group congruence  $\sim$ . We define a relation  $\approx$  on the semigroup  $\overline{\mathcal{B}}(G)$  as follows:

- (i)  $\left(\alpha_{(c,d)}^{(a,b)}, \alpha_{(r,s)}^{(p,q)}\right) \in \approx$  if and only if  $\left(\alpha_{(c,d)}^{(a,b)}, \alpha_{(r,s)}^{(p,q)}\right) \in \sim$ , for  $\alpha_{(c,d)}^{(a,b)}, \alpha_{(r,s)}^{(p,q)} \in \mathcal{B}(G) \subset \overline{\mathcal{B}}(G)$ ;
- (ii)  $\left(\alpha_{(r,s)}^{(p,q)}, \mathring{\alpha}_{(r,s)}^{(p,q)}\right), \left(\mathring{\alpha}_{(r,s)}^{(p,q)}, \alpha_{(r,s)}^{(p,q)}\right), \left(\mathring{\alpha}_{(r,s)}^{(p,q)}, \mathring{\alpha}_{(r,s)}^{(p,q)}\right) \in \approx$ , for all  $p, r \in A$  and  $q, s \in H$ ;



- (iii)  $(\alpha_{(c,d)}^{(a,b)}, \alpha_{(r,s)}^{(p,q)}) \in \approx$  if and only if  $(\alpha_{(c,d)}^{(a,b)}, \alpha_{(r,s)}^{(p,q)}) \in \sim$ , for  $\alpha_{(c,d)}^{(a,b)}, \alpha_{(r,s)}^{(p,q)} \in \mathcal{B}(G) \subset \overline{\mathcal{B}}(G)$  and  $\alpha_{(c,d)}^{(a,b)}, \alpha_{(r,s)}^{(p,q)} \in \mathring{\mathcal{B}}(G) \subset \overline{\mathcal{B}}(G)$ ;
- (iv)  $(\alpha_{(c,d)}^{(a,b)}, \alpha_{(r,s)}^{(p,q)}) \in \approx$  if and only if  $(\alpha_{(c,d)}^{(a,b)}, \alpha_{(r,s)}^{(p,q)}) \in \sim$ , for  $\alpha_{(c,d)}^{(a,b)}, \alpha_{(r,s)}^{(p,q)} \in \mathcal{B}(G) \subset \overline{\mathcal{B}}(G)$  and  $\alpha_{(c,d)}^{(a,b)} \in \mathring{\mathcal{B}}(G) \subset \overline{\mathcal{B}}(G)$ ;
- (v)  $(\alpha_{(c,d)}^{(a,b)}, \alpha_{(r,s)}^{(p,q)}) \in \approx$  if and only if  $(\alpha_{(c,d)}^{(a,b)}, \alpha_{(r,s)}^{(p,q)}) \in \sim$ , for  $\alpha_{(c,d)}^{(a,b)}, \alpha_{(r,s)}^{(p,q)} \in \mathcal{B}(G) \subset \overline{\mathcal{B}}(G)$  and  $\alpha_{(r,s)}^{(p,q)} \in \mathring{\mathcal{B}}(G) \subset \overline{\mathcal{B}}(G)$ .

Then simple verifications show that  $\approx$  is a congruence on the semigroup  $\overline{\mathcal{B}}(G)$ , and moreover the quotient semigroup  $\overline{\mathcal{B}}(G)/\approx$  is isomorphic to the quotient semigroup  $\mathcal{B}(G)/\sim$ . This implies that the congruence  $\approx$  is different from  $\sim_{\text{id}}$ . This contradicts that  $\sim_{\text{id}}$  is a unique non-trivial non-group congruence on the semigroup  $\overline{\mathcal{B}}(G)$ . The obtained contradiction implies implication (iv)  $\Rightarrow$  (i).

The proof of implication (v)  $\Rightarrow$  (i) is similar to implication (iv)  $\Rightarrow$  (i).  $\square$

**Theorem 3.10.** *Let  $G$  be a linearly ordered group and  $\mathfrak{C}_{mg}$  be the least group congruence on the semigroup  $\mathcal{B}(G)$  (resp.,  $\mathcal{B}^+(G)$ ). Then the quotient semigroup  $\mathcal{B}(G)/\mathfrak{C}_{mg}$  (resp.,  $\mathcal{B}^+(G)/\mathfrak{C}_{mg}$ ) is antiisomorphic to the group  $G$ .*

*Proof.* By Proposition 1.2(ii) and Lemma III.5.2 from [20] we have that elements  $\alpha_b^a$  and  $\alpha_d^c$  are  $\mathfrak{C}_{mg}$ -equivalent in  $\mathcal{B}(G)$  (resp., in  $\mathcal{B}^+(G)$ ) if and only if there exists  $x \in G$  such that  $\alpha_b^a \cdot \alpha_x^x = \alpha_d^c \cdot \alpha_x^x$ . Then Proposition 2.1(i) implies that  $\alpha_b^a \cdot \alpha_g^g = \alpha_d^c \cdot \alpha_g^g$  for all  $g \in G$  such that  $g \geq x$  in  $G$ . If  $g \geq b$  and  $g \geq d$  then the definition of the semigroup operation in  $\mathcal{B}(G)$  (resp., in  $\mathcal{B}^+(G)$ ) implies that  $\alpha_b^a \cdot \alpha_g^g = \alpha_g^{g \cdot b^{-1} \cdot a}$  and  $\alpha_d^c \cdot \alpha_g^g = \alpha_g^{g \cdot d^{-1} \cdot c}$ , and since  $G$  is a group we get that  $b^{-1} \cdot a = d^{-1} \cdot c$ .

Conversely, suppose that  $\alpha_b^a$  and  $\alpha_d^c$  are elements of the semigroup  $\mathcal{B}(G)$  (resp., in  $\mathcal{B}^+(G)$ ) such that  $b^{-1} \cdot a = d^{-1} \cdot c$ . Then for any element  $g \in G$  such that  $g \geq b$  and  $g \geq d$  in  $G$  we have that  $\alpha_b^a \cdot \alpha_g^g = \alpha_g^{g \cdot b^{-1} \cdot a}$  and  $\alpha_d^c \cdot \alpha_g^g = \alpha_g^{g \cdot d^{-1} \cdot c}$ , and hence since  $b^{-1} \cdot a = d^{-1} \cdot c$  we get that  $\alpha_b^a \mathfrak{C}_{mg} \alpha_d^c$ . Therefore,  $\alpha_b^a \mathfrak{C}_{mg} \alpha_d^c$  in  $\mathcal{B}(G)$  (resp., in  $\mathcal{B}^+(G)$ ) if and only if  $b^{-1} \cdot a = d^{-1} \cdot c$ .

We determine a map  $\mathfrak{f}: \mathcal{B}(G) \rightarrow G$  (resp.,  $\mathfrak{f}: \mathcal{B}^+(G) \rightarrow G$ ) by the formula  $(\alpha_b^a)\mathfrak{f} = b^{-1} \cdot a$ , for  $a, b \in G$ . Then we have that

$$\begin{aligned}
 (\alpha_b^a \cdot \alpha_d^c)\mathfrak{f} &= \begin{cases} (\alpha_d^{c \cdot b^{-1} \cdot a})\mathfrak{f}, & \text{if } b < c; \\ (\alpha_d^a)\mathfrak{f}, & \text{if } b = c; \\ (\alpha_{b \cdot c^{-1} \cdot d}^a)\mathfrak{f}, & \text{if } b > c, \end{cases} = \begin{cases} d^{-1} \cdot c \cdot b^{-1} \cdot a, & \text{if } b < c; \\ d^{-1} \cdot a, & \text{if } b = c; \\ (b \cdot c^{-1} \cdot d)^{-1} \cdot a, & \text{if } b > c, \end{cases} = \\
 &= \begin{cases} d^{-1} \cdot c \cdot b^{-1} \cdot a, & \text{if } b < c; \\ d^{-1} \cdot c \cdot b^{-1} \cdot a, & \text{if } b = c; \\ d^{-1} \cdot c \cdot b^{-1} \cdot a, & \text{if } b > c, \end{cases} = d^{-1} \cdot c \cdot b^{-1} \cdot a = (\alpha_d^c)\mathfrak{f} \cdot (\alpha_b^a)\mathfrak{f},
 \end{aligned}$$

for  $a, b, c, d \in G$ . This completes the proof of the theorem.  $\square$

Hölder's Theorem and Theorem 3.10 imply the following:

**Theorem 3.11.** *Let  $G$  be an archimedean linearly ordered group and  $\mathfrak{C}_{mg}$  be the least group congruence on the semigroup  $\mathcal{B}(G)$  (resp.,  $\mathcal{B}^+(G)$ ). Then the quotient semigroup  $\mathcal{B}(G)/\mathfrak{C}_{mg}$  (resp.,  $\mathcal{B}^+(G)/\mathfrak{C}_{mg}$ ) is isomorphic to the group  $G$ .*

Theorems 3.2, 3.3 and 3.11 imply the following:

**Corollary 3.12.** *Let  $G$  be an archimedean linearly ordered group and  $\mathfrak{C}_{mg}$  be the least group congruence on the semigroup  $\mathcal{B}(G)$  (resp.,  $\mathcal{B}^+(G)$ ). Then every non-isomorphic image of the semigroup  $\mathcal{B}(G)$  (resp.,  $\mathcal{B}^+(G)$ ) is isomorphic to some homomorphic image of the group  $G$ .*

**Theorem 3.13.** *Let  $G$  be a linearly ordered  $d$ -group and  $\mathfrak{C}_{mg}$  be the least group congruence on the semigroup  $\overline{\mathcal{B}}(G)$  (resp.,  $\overline{\mathcal{B}}^+(G)$ ). Then the quotient semigroup  $\overline{\mathcal{B}}(G)/\mathfrak{C}_{mg}$  (resp.,  $\overline{\mathcal{B}}^+(G)/\mathfrak{C}_{mg}$ ) is antiisomorphic to the group  $G$ .*

*Proof.* Similar arguments as in the proofs of Theorem 3.10 and Proposition 3.7 show that the following assertions are equivalent:

- (i)  $\alpha_b^a \mathfrak{C}_{mg} \alpha_d^c$  in  $\overline{\mathcal{B}}(G)$  (resp., in  $\overline{\mathcal{B}}^+(G)$ );
- (ii)  $\alpha_b^a \mathfrak{C}_{mg} \hat{\alpha}_d^c$  in  $\overline{\mathcal{B}}(G)$  (resp., in  $\overline{\mathcal{B}}^+(G)$ );
- (iii)  $\hat{\alpha}_b^a \mathfrak{C}_{mg} \hat{\alpha}_d^c$  in  $\overline{\mathcal{B}}(G)$  (resp., in  $\overline{\mathcal{B}}^+(G)$ );
- (iv)  $b^{-1} \cdot a = d^{-1} \cdot c$ .

We determine a map  $\mathfrak{f}: \mathcal{B}(G) \rightarrow G$  (resp.,  $\mathfrak{f}: \mathcal{B}^+(G) \rightarrow G$ ) by the formulae  $(\alpha_b^a)\mathfrak{f} = b^{-1} \cdot a$  and  $(\hat{\alpha}_b^a)\mathfrak{f} = b^{-1} \cdot a$ , for  $a, b \in G$ . Then we have that

$$\begin{aligned}
 (\alpha_b^a \cdot \alpha_d^c)\mathfrak{f} &= (\alpha_d^c)\mathfrak{f} \cdot (\alpha_b^a)\mathfrak{f}, \\
 (\hat{\alpha}_b^a \cdot \hat{\alpha}_d^c)\mathfrak{f} &= \begin{cases} (\hat{\alpha}_d^{c \cdot b^{-1} \cdot a})\mathfrak{f}, & \text{if } b < c; \\ (\hat{\alpha}_d^a)\mathfrak{f}, & \text{if } b = c; \\ (\hat{\alpha}_{b \cdot c^{-1} \cdot d}^a)\mathfrak{f}, & \text{if } b > c, \end{cases} = \begin{cases} d^{-1} \cdot c \cdot b^{-1} \cdot a, & \text{if } b < c; \\ d^{-1} \cdot a, & \text{if } b = c; \\ (b \cdot c^{-1} \cdot d)^{-1} \cdot a, & \text{if } b > c, \end{cases} = \\
 &= \begin{cases} d^{-1} \cdot c \cdot b^{-1} \cdot a, & \text{if } b < c; \\ d^{-1} \cdot c \cdot b^{-1} \cdot a, & \text{if } b = c; \\ d^{-1} \cdot c \cdot b^{-1} \cdot a, & \text{if } b > c, \end{cases} = d^{-1} \cdot c \cdot b^{-1} \cdot a = (\hat{\alpha}_d^c)\mathfrak{f} \cdot (\hat{\alpha}_b^a)\mathfrak{f}, \\
 (\alpha_b^a \cdot \hat{\alpha}_d^c)\mathfrak{f} &= \begin{cases} (\hat{\alpha}_d^{c \cdot b^{-1} \cdot a})\mathfrak{f}, & \text{if } b < c; \\ (\hat{\alpha}_d^a)\mathfrak{f}, & \text{if } b = c; \\ (\alpha_{b \cdot c^{-1} \cdot d}^a)\mathfrak{f}, & \text{if } b > c, \end{cases} = \begin{cases} d^{-1} \cdot c \cdot b^{-1} \cdot a, & \text{if } b < c; \\ d^{-1} \cdot a, & \text{if } b = c; \\ (b \cdot c^{-1} \cdot d)^{-1} \cdot a, & \text{if } b > c, \end{cases} = \\
 &= \begin{cases} d^{-1} \cdot c \cdot b^{-1} \cdot a, & \text{if } b < c; \\ d^{-1} \cdot c \cdot b^{-1} \cdot a, & \text{if } b = c; \\ d^{-1} \cdot c \cdot b^{-1} \cdot a, & \text{if } b > c, \end{cases} = d^{-1} \cdot c \cdot b^{-1} \cdot a = (\alpha_d^c)\mathfrak{f} \cdot (\hat{\alpha}_b^a)\mathfrak{f},
 \end{aligned}$$

$$\begin{aligned}
(\alpha_b^a \cdot \alpha_d^c) \mathfrak{f} &= \begin{cases} (\alpha_d^{c \cdot b^{-1} \cdot a}) \mathfrak{f}, & \text{if } b < c; \\ (\alpha_d^a) \mathfrak{f}, & \text{if } b = c; \\ (\alpha_{b \cdot c^{-1} \cdot d}^a) \mathfrak{f}, & \text{if } b > c, \end{cases} = \begin{cases} d^{-1} \cdot c \cdot b^{-1} \cdot a, & \text{if } b < c; \\ d^{-1} \cdot a, & \text{if } b = c; \\ (b \cdot c^{-1} \cdot d)^{-1} \cdot a, & \text{if } b > c, \end{cases} = \\
&= \begin{cases} d^{-1} \cdot c \cdot b^{-1} \cdot a, & \text{if } b < c; \\ d^{-1} \cdot c \cdot b^{-1} \cdot a, & \text{if } b = c; \\ d^{-1} \cdot c \cdot b^{-1} \cdot a, & \text{if } b > c, \end{cases} = d^{-1} \cdot c \cdot b^{-1} \cdot a = (\alpha_d^c) \mathfrak{f} \cdot (\alpha_b^a) \mathfrak{f},
\end{aligned}$$

for  $a, b, c, d \in G$ . This completes the proof of the theorem.  $\square$

Hölder's Theorem and Theorem 3.13 imply the following:

**Theorem 3.14.** *Let  $G$  be an archimedean linearly ordered  $d$ -group and  $\mathfrak{C}_{mg}$  be the least group congruence on the semigroup  $\overline{\mathcal{B}}(G)$  (resp.,  $\overline{\mathcal{B}}^+(G)$ ). Then the quotient semigroup  $\overline{\mathcal{B}}(G)/\mathfrak{C}_{mg}$  (resp.,  $\overline{\mathcal{B}}^+(G)/\mathfrak{C}_{mg}$ ) is isomorphic to the group  $G$ .*

Theorems 3.8 and 3.14 imply the following:

**Corollary 3.15.** *Let  $G$  be an archimedean linearly ordered  $d$ -group,  $T$  be a semigroup and  $h: \overline{\mathcal{B}}(G) \rightarrow T$  (resp.,  $h: \overline{\mathcal{B}}^+(G) \rightarrow T$ ) be a homomorphism. Then only one of the following conditions holds:*

- (i)  $h$  is a monomorphism;
- (ii) the image  $(\overline{\mathcal{B}}(G)) h$  (resp.,  $(\overline{\mathcal{B}}^+(G)) h$ ) is isomorphic to some homomorphic image of the group  $G$ ;
- (iii) the image  $(\overline{\mathcal{B}}(G)) h$  (resp.,  $(\overline{\mathcal{B}}^+(G)) h$ ) is isomorphic to the semigroup  $\mathcal{B}(G)$  (resp.,  $\mathcal{B}^+(G)$ ).

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